

Turbulent Two-Dimensional Jet

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Turbulent Flow

- ▶ Turbulent Flow: Irregular or random fluctuation (mixing or rotational motion) that is superimposed on the mainstream
- ▶ Feature of fluid flow and not of a fluid
- ▶ Turbulent flow = mean flow + random fluctuations with zero mean

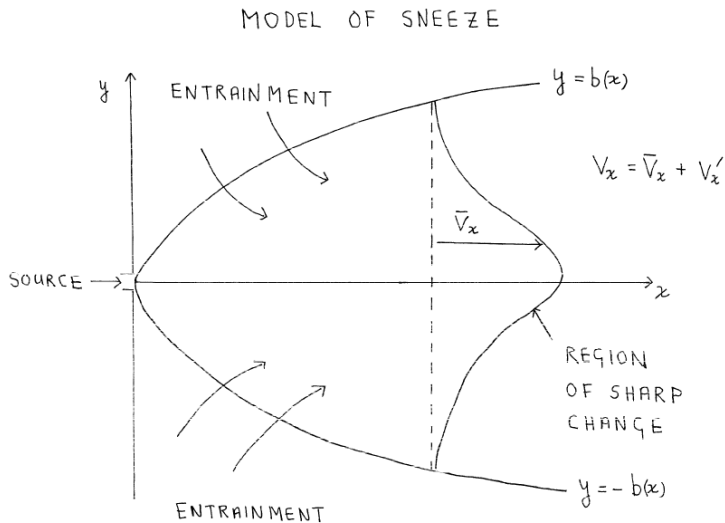
$$V_i(x, y, t) = \bar{V}_i(x, y) + V'_i(x, y, t) \quad (1)$$

$$\overline{\bar{V}_i} = \bar{V}_i, \overline{V'_i} = 0, \overline{V'_i V'_j} \neq 0$$

Aims and Objectives

- ▶ Model of sneeze
- ▶ Fit parameter values for a sneeze
- ▶ Range of jet in downstream x-direction
- ▶ Range of jet in transverse y-direction
- ▶ Estimates of Social Distancing
- ▶ Model effect of mask by changing value of J

Model of a Sneeze



Governing Equations

- Reynolds averaged equations in the boundary layer approximation:

$$\bar{V}_x \frac{\partial \bar{V}_x}{\partial x} + \bar{V}_y \frac{\partial \bar{V}_x}{\partial y} = \frac{\partial}{\partial y} \left[(\nu + \nu_T) \frac{\partial \bar{V}_x}{\partial y} \right] \quad (2)$$

$$\nu_T = \ell^2(x) \left| \frac{\partial \bar{V}_x}{\partial y} \right| \quad (3)$$

- Conservation of Mass:

$$\frac{\partial \bar{V}_x}{\partial x} + \frac{\partial \bar{V}_y}{\partial y} = 0 \quad (4)$$

Two-Dimensional Boundary Layer Equation

- The two-dimensional boundary equation is:

$$\frac{\partial \Psi}{\partial y} \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x} \frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial}{\partial y} \left[\left(\nu - \ell^2(x) \frac{\partial^2 \Psi}{\partial y^2} \right) \frac{\partial^2 \Psi}{\partial y^2} \right], \quad (5)$$

$$\bar{V}_x = \frac{\partial \Psi}{\partial y}, \quad \bar{V}_y = -\frac{\partial \Psi}{\partial x}$$

Proving Invariance

- Scaling transformations:

$$\bar{x} = \lambda^a x, \bar{y} = \lambda^b y, \bar{\Psi} = \lambda^c \Psi, \bar{\ell} = \lambda^m \ell \quad (6)$$

- Equation (5) is invariant,

$$\frac{\partial \bar{\Psi}}{\partial \bar{y}} \frac{\partial^2 \bar{\Psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\Psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\Psi}}{\partial \bar{y}^2} = \frac{\partial}{\partial \bar{y}} \left[\left(\nu - \bar{\ell}^2(\bar{x}) \frac{\partial^2 \bar{\Psi}}{\partial \bar{y}^2} \right) \frac{\partial^2 \bar{\Psi}}{\partial \bar{y}^2} \right] \quad (7)$$

- Provided

$$c = a - b, \quad m = \frac{1}{2}(3b - a) \quad (8)$$

General Solution of Streamline Function

- ▶ Suppose (5) has a general solution:

$$\Psi = f(x, y) \quad (9)$$

- ▶ With corresponding invariant solution:

$$\overline{\Psi} = f(\overline{x}, \overline{y}) \quad (10)$$

- ▶ Streamline function:

$$\Psi(x, y) = x^{\beta} F(\xi), \quad \beta = \frac{c}{a} \quad (11)$$

$$\xi = yx^{-\alpha}, \quad \alpha = \frac{b}{a} \quad (12)$$

General solution of Prandtl's Mixing Length

- Suppose a solution of the form:

$$\ell = h(x) \quad (13)$$

- With corresponding invariant solution:

$$\bar{\ell} = h(\bar{x}) \quad (14)$$

- Prandtl's mixing length:

$$\ell(x) = \ell_0 x^{\frac{m}{a}} \quad (15)$$

- Note that ℓ_0 is a constant

Momentum Flux

- Given by

$$J = 2\rho \int_0^{b(x)} \bar{V}_x^2(x, y) dy \quad (16)$$

$$= 2\rho \int_0^{b(x)} \left(\frac{\partial \Psi}{\partial y} \right)^2 dy \quad (17)$$

- Since $J = \text{constant independent of } x$:

$$\alpha = 2\beta, \quad (18)$$

$$b(x) = \xi_b x^\alpha \quad (19)$$

- Note that ξ_b is a constant

Case 1: $\nu \neq 0, \alpha = \frac{2}{3}$

- ▶ Three unknowns, two equations
- ▶ Parameter values:

$$\alpha = \frac{2}{3}, \beta = \frac{1}{3}, \frac{m}{a} = \frac{1}{2} \quad (20)$$

- ▶ Hence,

$$\Psi(x, y) = x^{\frac{1}{3}} F(\xi), \quad \xi = yx^{-\frac{2}{3}}, \quad \ell(x) = \ell_0 x^{\frac{1}{2}}, \quad b(x) = \xi_b x^{\frac{2}{3}} \quad (21)$$

- ▶ Prandtl's hypothesis is not satisfied

Case 2: $\nu = 0$, $\alpha = 1$

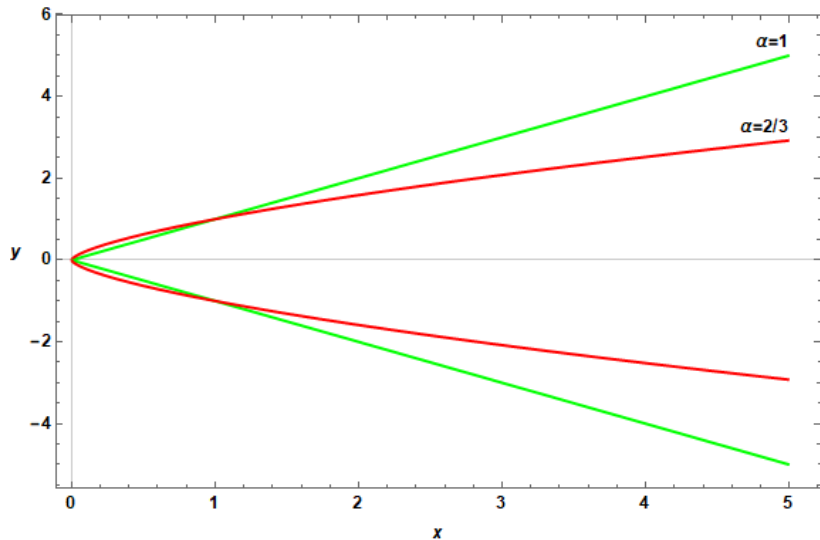
- ▶ Three unknowns, one equation
- ▶ Satisfying Prandtl's hypothesis

$$\alpha = 1 \quad (22)$$

- ▶ Hence,

$$\Psi(x, y) = x^{\frac{1}{2}} F(\xi), \quad \xi = yx^{-1}, \quad \ell(x) = \ell_0 x, \quad b(x) = \xi_b x \quad (23)$$

Plot of the Boundary Equation $b(x)$ for Different Parameters of α



Velocities

- ▶ ODE must be independent of x and y . Rewrite similarity solution i.t.o ξ and x

$$y = \xi x^\alpha \quad (24)$$

$$\begin{aligned} V_x(x, y) &= \frac{\partial \Psi}{\partial y} \\ &= x^{\frac{-\alpha}{2}} \frac{dF}{d\xi} \end{aligned} \quad (25)$$

$$\begin{aligned} V_y(x, y) &= -\frac{\partial \Psi}{\partial x} \\ &= \frac{\alpha}{2} x^{\frac{\alpha}{2}-1} \left[-F(\xi) + 2\xi \frac{dF}{d\xi} \right] \end{aligned} \quad (26)$$

Boundary Conditions

$$\frac{\partial \bar{V}_x}{\partial y}(x, 0) = 0 \text{ (Turning point)} \quad (27)$$

$$\bar{V}_y(x, 0) = 0 \text{ (No cavities)} \quad (28)$$

$$\frac{\partial \bar{V}_x}{\partial y}(x, b(x)) = 0 \text{ (Zero kinematic eddy viscosity)} \quad (29)$$

$$\bar{V}_x(x, b(x)) = 0 \text{ (Limit tends to zero)} \quad (30)$$

Ordinary Differential Equations (ODEs)

- $\nu \neq 0, \alpha = \frac{2}{3}$:

$$\ell_0^2 \left(\frac{d^2 F}{d\xi^2} \right)^2 - \nu \frac{d^2 F}{d\xi^2} - \frac{1}{3} F \frac{dF}{d\xi} = 0 \quad (31)$$

- $\nu = 0, \alpha$ is arbitrary

$$\ell_0^2 \left(\frac{d^2 F}{d\xi^2} \right)^2 - \frac{\alpha}{2} F \frac{dF}{d\xi} = 0 \quad (32)$$

Analytical Solution for Case 2

- ▶ Equation (32) is solved analytically in order to determine the parameters for the velocity equations
- ▶ Find values for the constants, ξ_b , and K

Entrainment

- At the boundary $y = b(x) = \xi_b x^\alpha$:

$$\xi_b = \frac{4\pi}{3\sqrt{3}} \left(\frac{2\ell_0^2}{\alpha} \right)^{\frac{1}{3}} \quad (33)$$

Boundary of a Jet

- Determining the Value for K:

$$K = \left(\frac{J}{2\rho} \right)^{\frac{1}{2}} \left(\frac{2\ell_0^2}{\alpha} \right)^{\frac{1}{6}} \frac{\left[\Gamma\left(\frac{1}{3}\right) \right]^{\frac{1}{2}}}{\Gamma\left(\frac{2}{3}\right)} \quad (34)$$

- Entrainment:

$$v_y(x, b(x)) = -\frac{\alpha}{2} x \left(\frac{\alpha}{2}^{-1} \right) K \quad (35)$$

Velocity in the x -direction

- ▶ At the centerline of the Jet $y = 0$:
- ▶ Velocity

$$v_x(x, 0) = \left(\frac{\alpha}{2\ell_0^2} \right)^{\frac{1}{3}} Kx^{-\frac{\alpha}{2}} \quad (36)$$

Numerical Solution

- ▶ Equation (31) is solved numerically
- ▶ Quadratic form and making the second derivative the subject of the formula
- ▶ Shooting method
- ▶ Shooting towards a conserved quantity and not a derivative

Expectation

